Priority Queues (Heaps)

Advanced Data Structures

Motivation

- Queues are a standard mechanism for ordering tasks on a first-come, first-served basis
- However, some tasks may be more important or timely than others (higher priority)
- Priority queues
  - Store tasks using a partial ordering based on priority
  - Ensure highest priority task at head of queue
- Heaps are the underlying data structure of priority queues

Priority Queues

- Main operations
  - **`insert`** (i.e., enqueue)
  - **`deleteMin`** (i.e., dequeue)
    - Finds the minimum element in the queue, deletes it from the queue, and returns it
- Performance
  - Goal is for operations to be fast
  - Will be able to achieve $O(\log_2 N)$ time insert/deleteMin amortized over multiple operations
  - Will be able to achieve O(1) time inserts amortized over multiple insertions

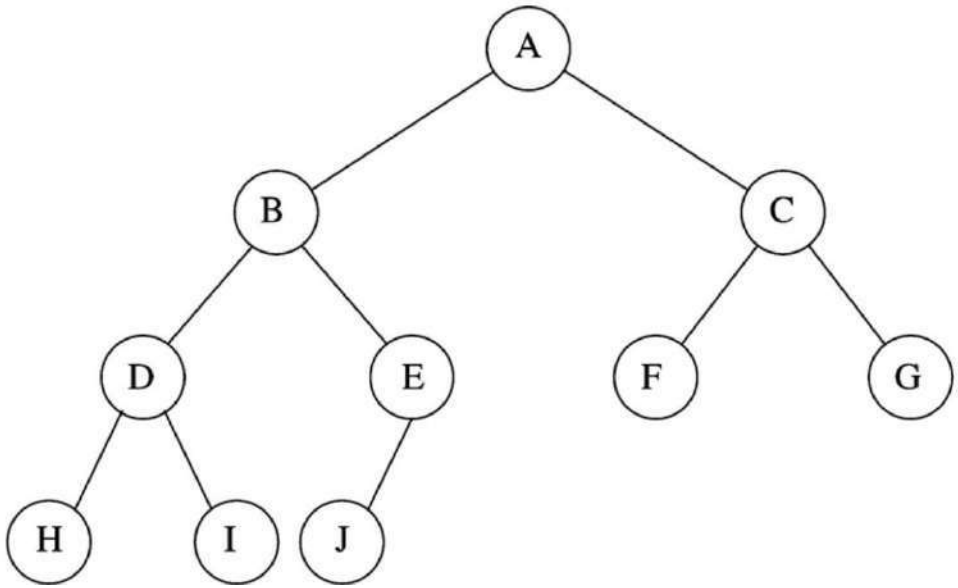
Simple Implementations

- Unordered list
  - O(1) insert
  - O(N) deleteMin
- Ordered list
  - O(N) insert
  - O(1) deleteMin
- Balanced BST
  - $O(\log_2 N)$ insert and deleteMin
- Observation: We don’t need to keep the priority queue completely ordered

Binary Heap

- A binary heap is a binary tree with two properties
- Structure property
  - A binary heap is a complete binary tree
    - Each level is completely filled
    - Bottom level may be partially filled from left to right
- Height of a complete binary tree with N elements is $\log_2 N$

Binary Heap Example



	A	B	C	D	E	F	G	H	I	J			
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Binary Heap

- Heap-order property
  - For every node X, key(parent(X)) ≤ key(X)
  - Except root node, which has no parent
- Thus, minimum key always at root
  - Or, maximum, if you choose
- Insert and deleteMin must maintain heap-order property

Implementing Complete Binary Trees as Arrays

- Given element at position i in the array
  - i’s left child is at position 2i
  - i’s right child is at position 2i+1
  - i’s parent is at position $i / 2$

```
template <typename Comparable>
class BinaryHeap
{
  public:
    explicit BinaryHeap( int capacity = 100 );
    explicit BinaryHeap( const vector<Comparable> & items );

    bool isEmpty( ) const;
    const Comparable & findMin( ) const;

    void insert( const Comparable & x );
    void deleteMin( );
    void deleteMin( Comparable & minItem );
    void makeEmpty( );

  private:
    int                currentSize;  // Number of elements in heap
    vector<Comparable> array;        // The heap array

    void buildHeap( );
    void percolateDown( int hole );
};
```

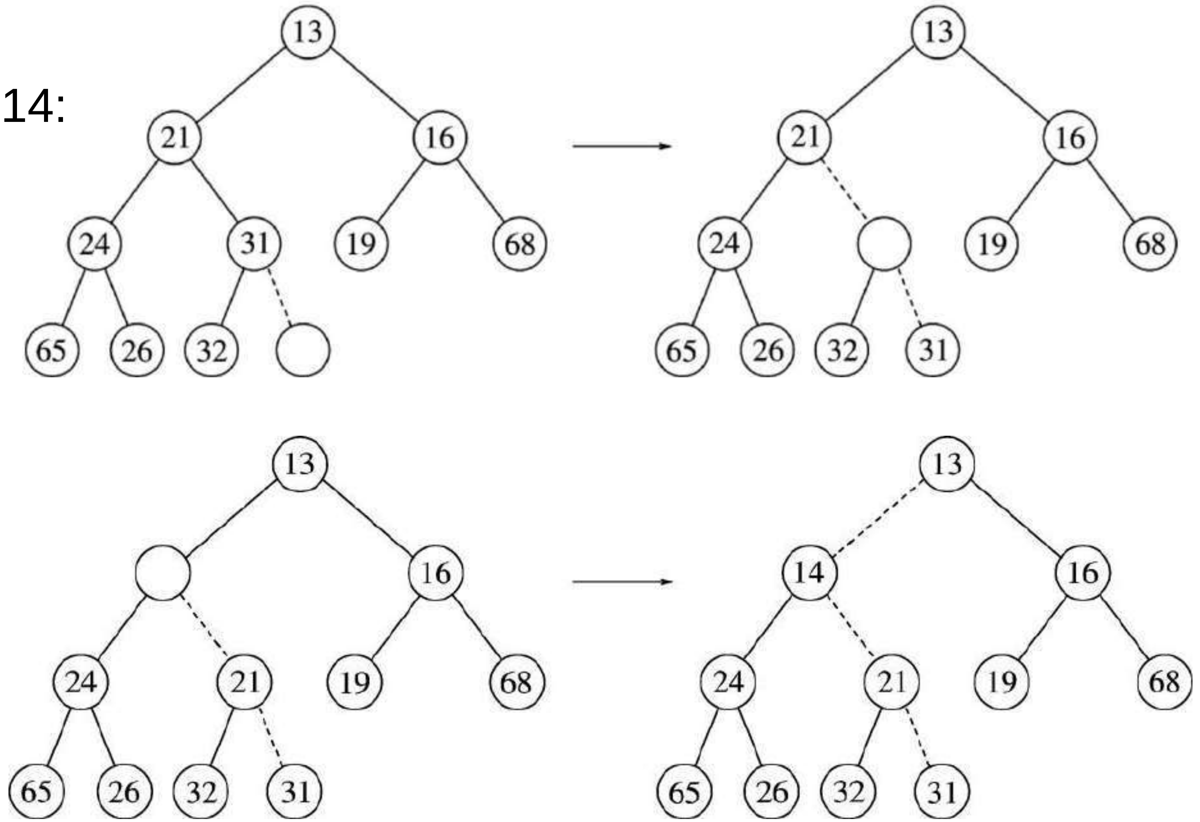
Fix heap after deleteMin

Heap Insert

- Insert new element into the heap at the next available slot (“hole”)
  - According to maintaining a complete binary tree
- Then, “percolate” the element up the heap while heap-order property not satisfied

Heap Insert: Example

Insert 14:



Heap Insert: Implementation

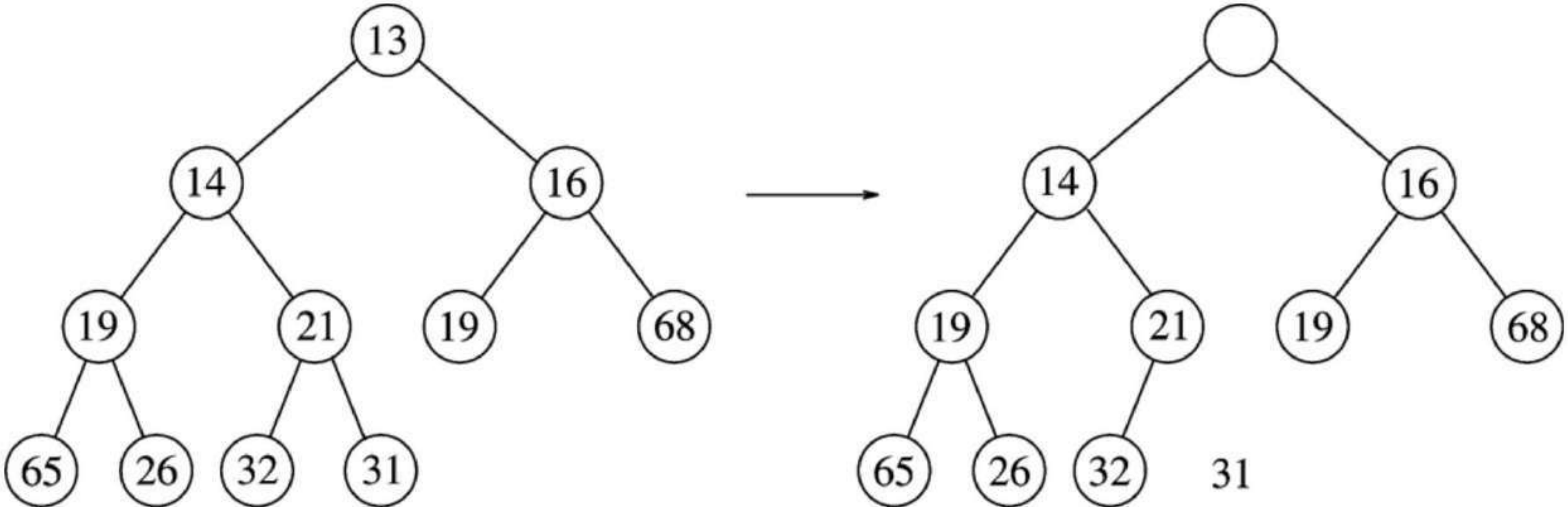
```
/**
 * Insert item x, allowing duplicates.
 */
void insert( const Comparable & x )
{
    if( currentSize == array.size( ) - 1 )
        array.resize( array.size( ) * 2 );

        // Percolate up
    int hole = ++currentSize;
    for( ; hole > 1 && x < array[ hole / 2 ]; hole /= 2 )
        array[ hole ] = array[ hole / 2 ];
    array[ hole ] = x;
}
```

1
2

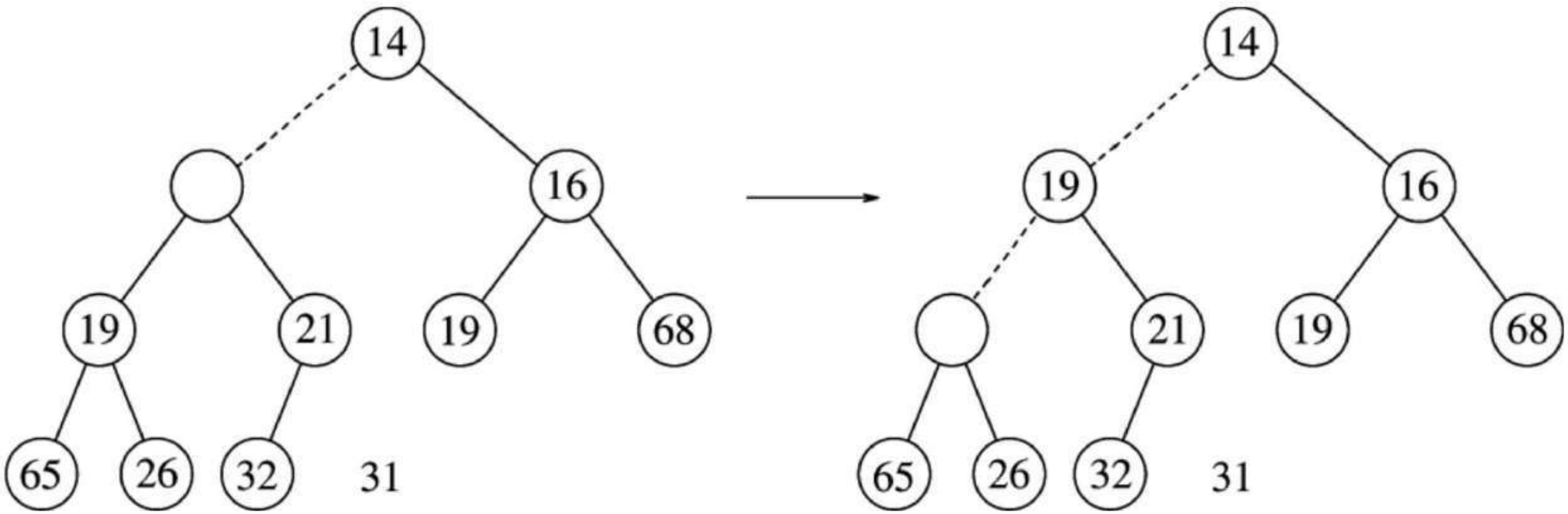
Heap DeleteMin

- Minimum element is always at the root
- Heap decreases by one in size
- Move last element into hole at root
- Percolate down while heap-order property not satisfied

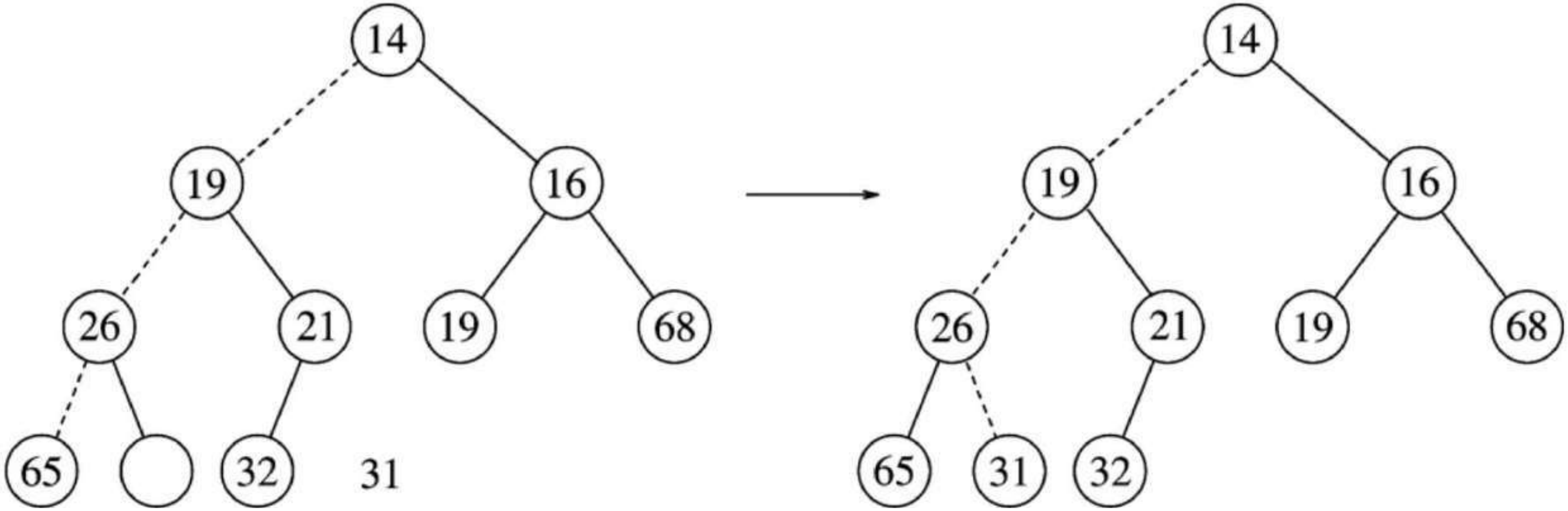
Heap DeleteMin: Example



Heap DeleteMin: Example



Heap DeleteMin: Example



Heap DeleteMin: Implementation

```
/**
 * Remove the minimum item.
 * Throws UnderflowException if empty.
 */
void deleteMin( )
{
    if( isEmpty( ) )
        throw UnderflowException( );

    array[ 1 ] = array[ currentSize-- ];
    percolateDown( 1 );
}
```

```
/**
 * Remove the minimum item and place it in minItem.
 * Throws UnderflowException if empty.
 */
void deleteMin( Comparable & minItem )
{
    if( isEmpty( ) )
        throw UnderflowException( );

    minItem = array[ 1 ];
    array[ 1 ] = array[ currentSize-- ];
    percolateDown( 1 );
}
```

Heap DeleteMin: Implementation

```
/**
 * Internal method to percolate down in the heap.
 * hole is the index at which the percolate begins.
 */
void percolateDown( int hole )
{
    int child;
    Comparable tmp = array[ hole ];

    for( ; hole * 2 <= currentSize; hole = child )
    {
        child = hole * 2;
        if( child != currentSize && array[ child + 1 ] < array[ child ] )
            child++;
        if( array[ child ] < tmp )
            array[ hole ] = array[ child ];
        else
            break;
    }
    array[ hole ] = tmp;
}
```

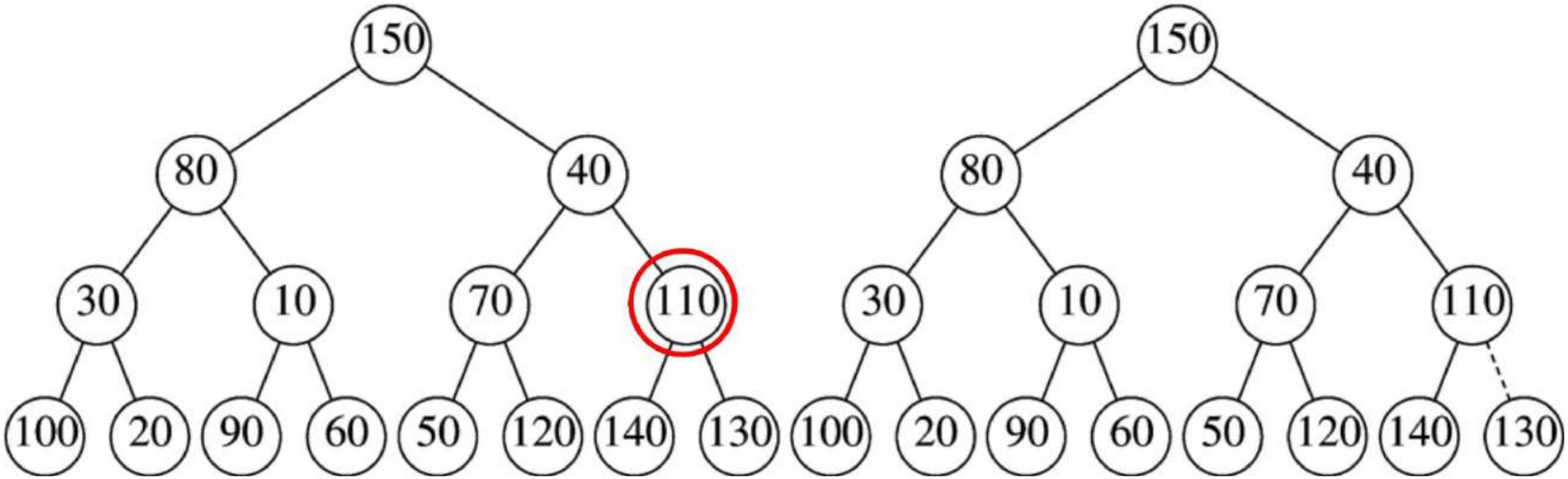
Other Heap Operations

- decreaseKey(p,v)
  - Lowers value of item p to v
  - Need to percolate up
  - E.g., change job priority
- increaseKey(p,v)
  - Increases value of item p to v
  - Need to percolate down
- remove(p)
  - First, decreaseKey(p,-∞)
  - Then, deleteMin
  - E.g., terminate job

Building a Heap

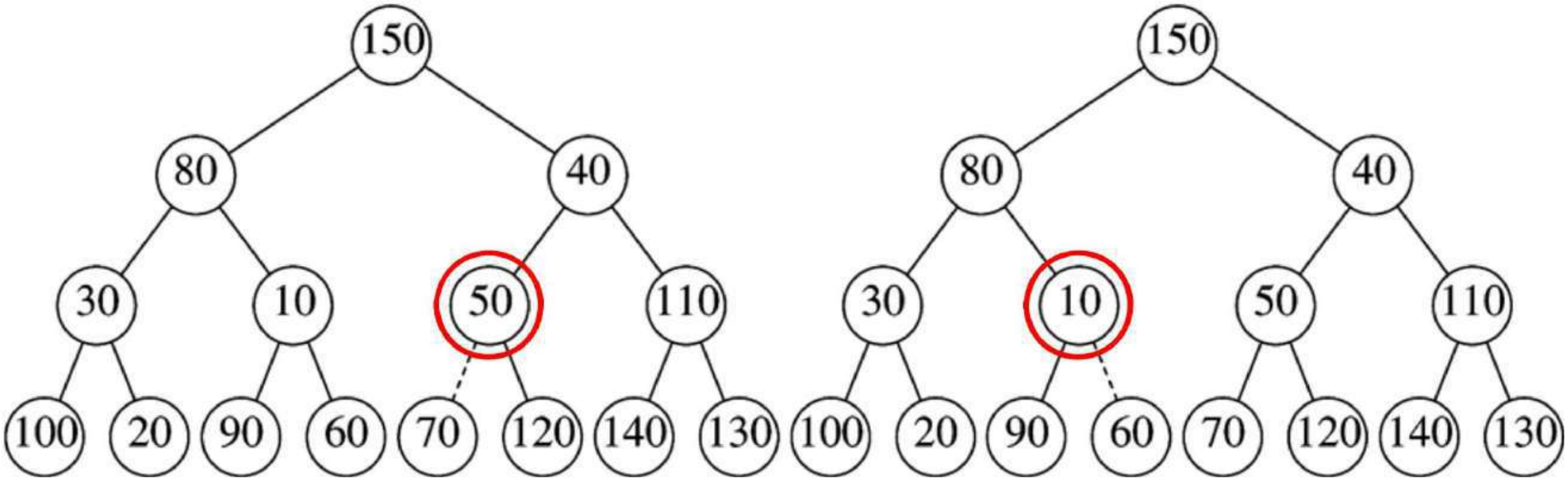
- Construct heap from initial set of N items
- Solution 1
  - Perform N inserts
  - O(N) average case, but O(N $\log_2$ N) worst-case
- Solution 2
  - Assume initial set is a heap
  - Perform a percolate-down from each internal node (H[size/2] to H[1])

BuildHeap Example

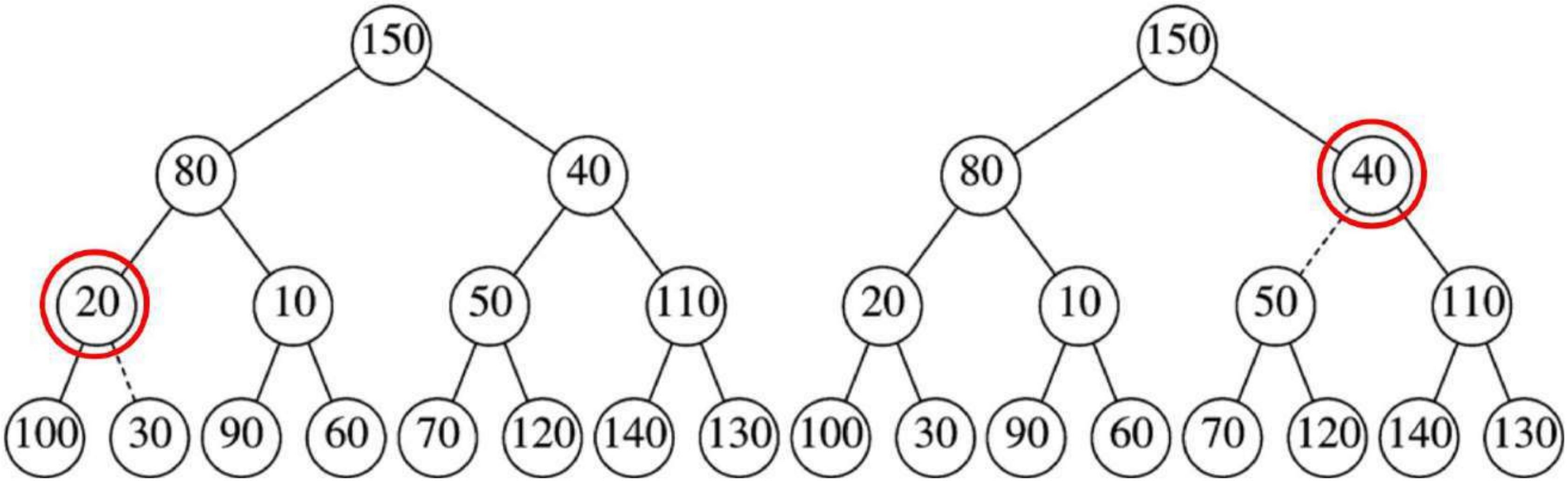


Leaves are all valid heaps

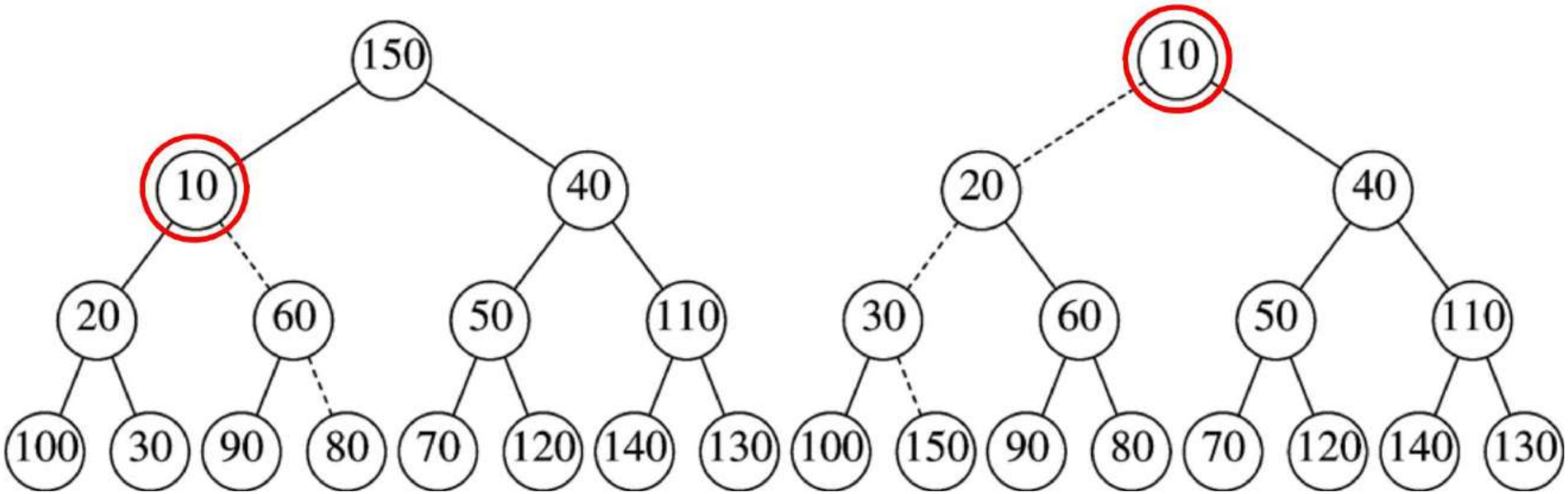
BuildHeap Example



BuildHeap Example



BuildHeap Example



BuildHeap Implementation

```
explicit BinaryHeap( const vector<Comparable> & items )
  : array( items.size( ) + 10 ), currentSize( items.size( ) )
{
    for( int i = 0; i < items.size( ); i++ )
        array[ i + 1 ] = items[ i ];
    buildHeap( );
}

/**
 * Establish heap order property from an arbitrary
 * arrangement of items. Runs in linear time.
 */
void buildHeap( )
{
    for( int i = currentSize / 2; i > 0; i-- )
        percolateDown( i );
}
```

BuildHeap Analysis

- Running time of buildHeap proportional to sum of the heights of the nodes
- Theorem 6.1
  - For the perfect binary tree of height h containing $2^{h+1} - 1$ nodes, the sum of heights of the nodes is $2^{h+1} - 1 - (h + 1)$
- Since $N = 2^{h+1} - 1$, then sum of heights is O(N)
- Slightly better for complete binary tree

Binary Heap Operations Worst-case Analysis

- Height of heap is $\log_2 N$
- insert: $O(\log_2 N)$
  - 2.607 comparisons on average, i.e., O(1)
- deleteMin: $O(\log_2 N)$
- decreaseKey: $O(\log_2 N)$
- increaseKey: $O(\log_2 N)$
- remove: $O(\log_2 N)$
- buildHeap: O(N)

Applications

- Operating system scheduling
  - Process jobs by priority
- Graph algorithms
  - Find the least-cost, neighboring vertex
- Event simulation
  - Instead of checking for events at each time click, look up next event to happen

Priority Queues: Alternatives to Binary Heaps

- d-Heap
  - Each node has d children
  - insert in $O(\log_d N)$ time
  - deleteMin in $O(d \log_d N)$ time
- Binary heaps are 2-Heaps

Mergeable Heaps

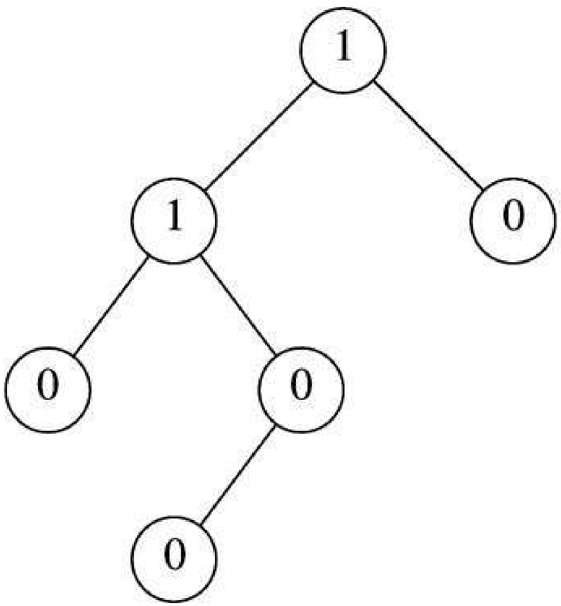
- Heap merge operation
  - Useful for many applications
  - Merge two (or more) heaps into one
  - Identify new minimum element
  - Maintain heap-order property
  - Merge in O(log N) time
  - Still support insert and deleteMin in O(log N) time
    - Insert = merge existing heap with one-element heap
- d-Heaps require O(N) time to merge

Leftist Heaps

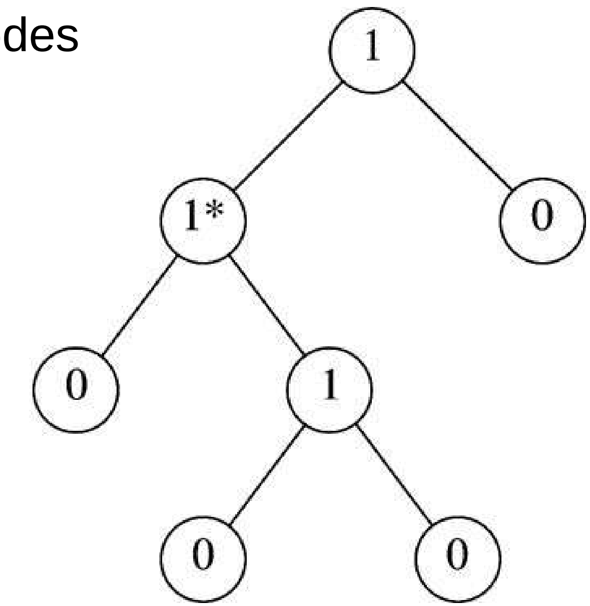
- Null path length npl(X) of node X
  - Length of the shortest path from X to a node without two children
- Leftist heap property
  - For every node X in heap, npl(leftChild(X)) ≥ npl(rightChild(X))
- Leftist heaps have deep left subtrees and shallow right subtrees
  - Thus if operations reside in right subtree, they will be faster

Leftist Heaps

npl(X) shown in nodes



Leftist heap



Not a leftist heap

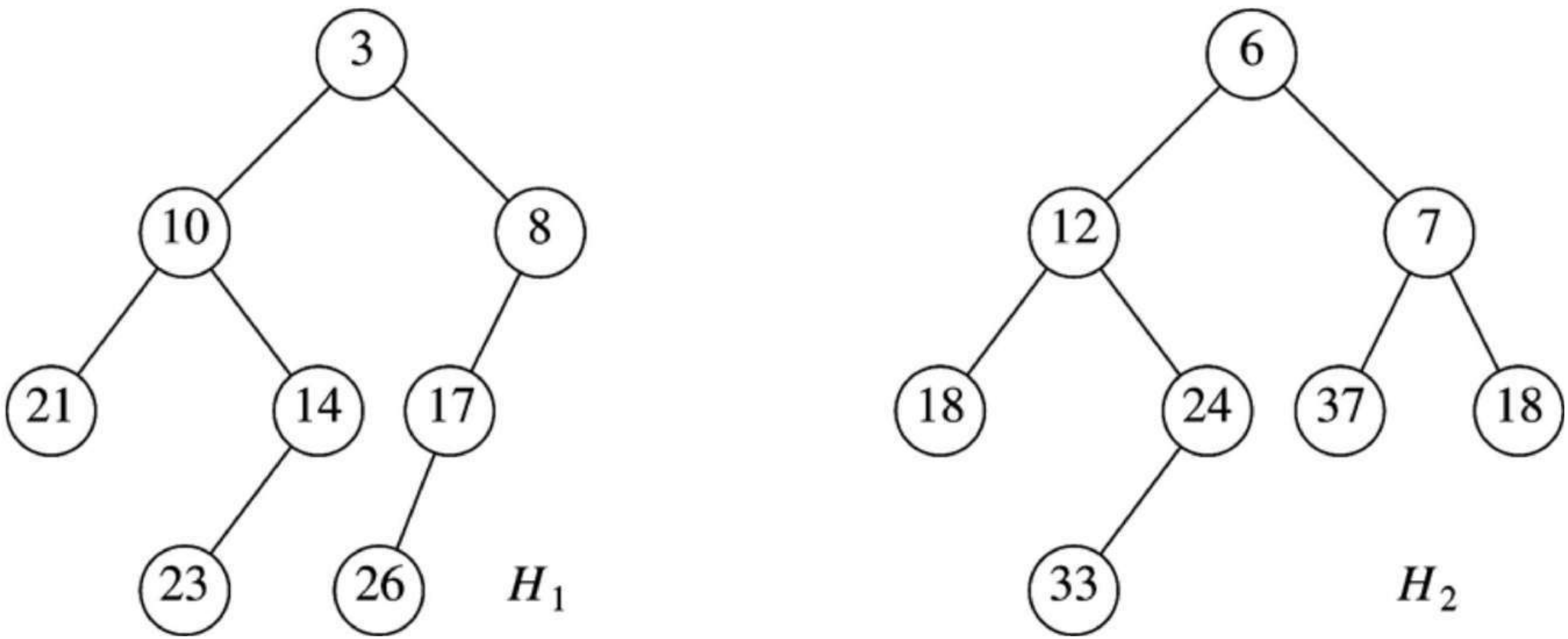
Leftist Heaps

- Theorem 6.2
  - A leftist tree with r nodes on the right path must have at least $2^r - 1$ nodes.
- Thus, a leftist tree with N nodes has a right path with at most $\log(N+1)$ nodes

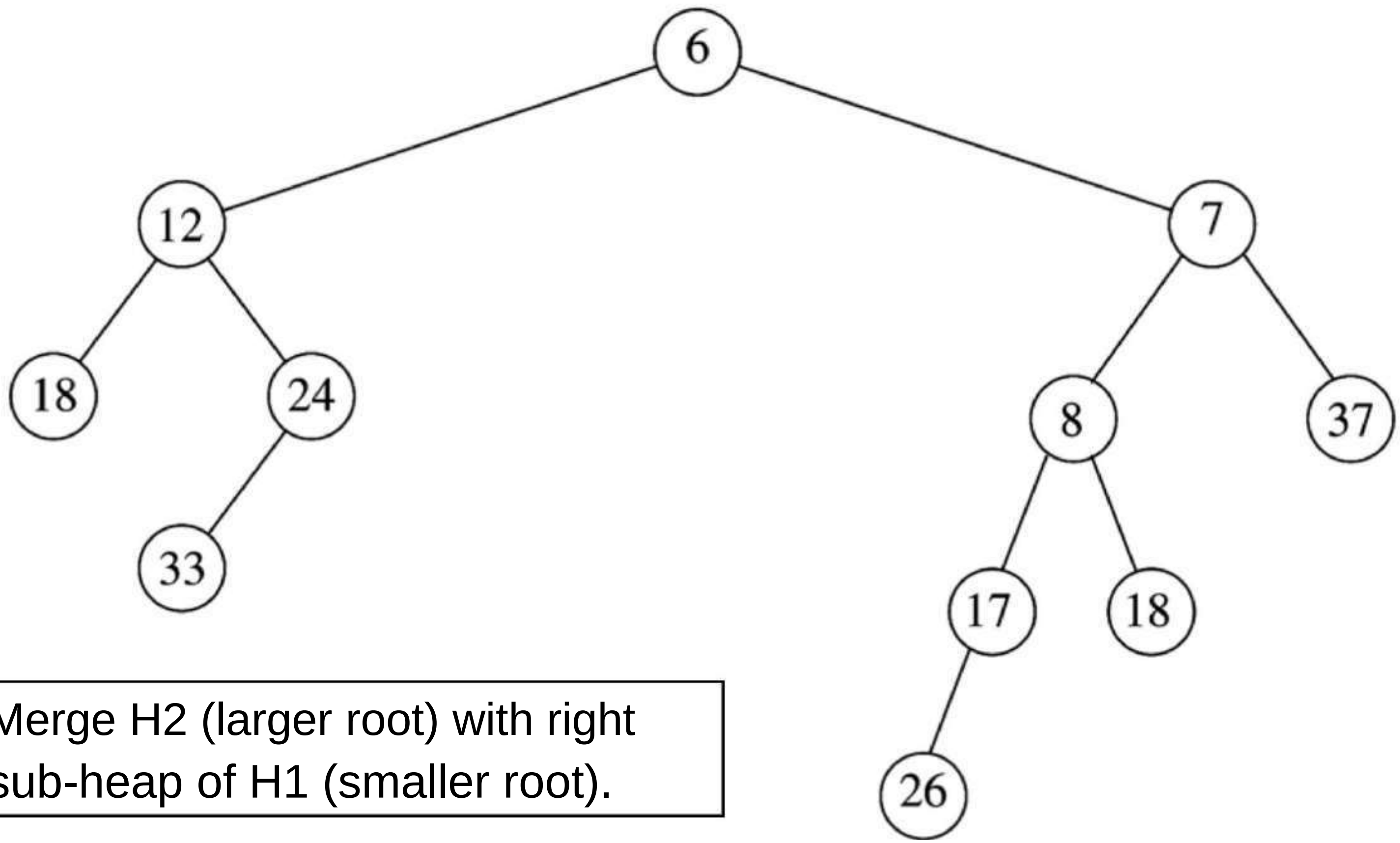
Leftist Heaps

- Merge heaps H1 and H2
  - Assume root(H1) > root(H2)
  - Recursively merge H1 with right subheap of H2
  - If result is not leftist, then swap the left and right subheaps
  - Running time O(log N)
- DeleteMin
  - Delete root and merge children

Leftist Heaps: Example

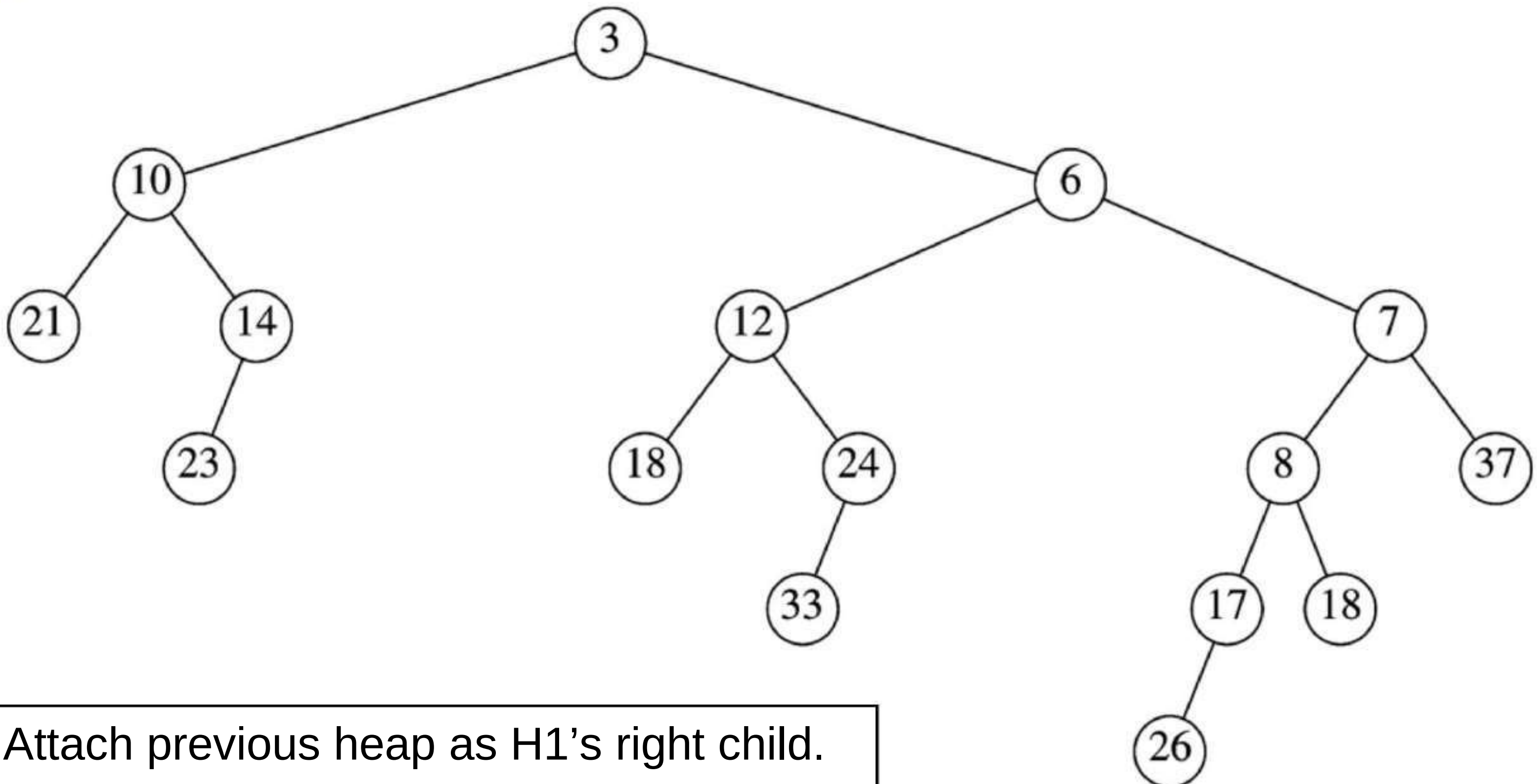


Leftist Heaps: Example



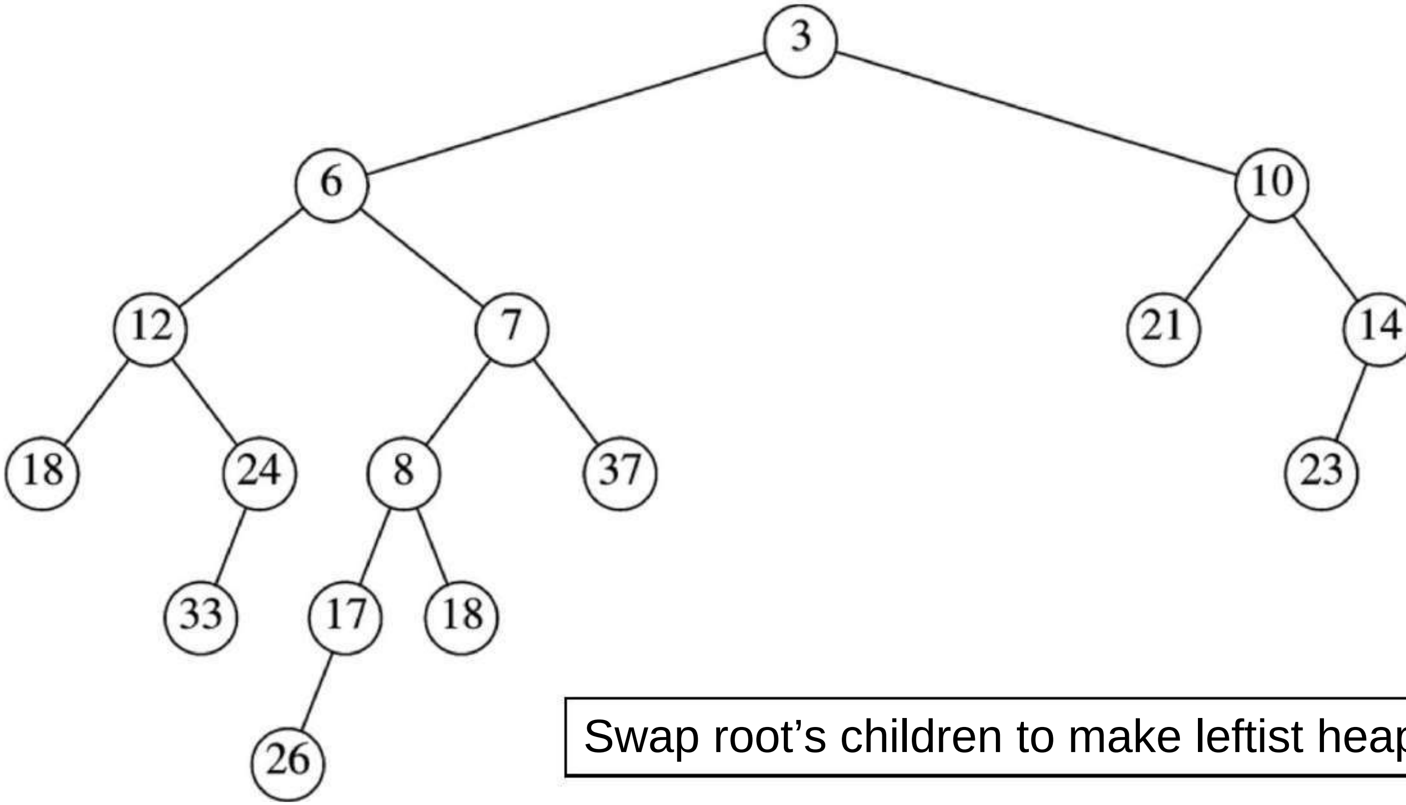
Merge H2 (larger root) with right sub-heap of H1 (smaller root).

Leftist Heaps: Example



Attach previous heap as H1’s right child.
Leftist heap?

Leftist Heaps: Example



Swap root's children to make leftist heap.

Skew Heaps

- Self-adjusting version of leftist heap
- Skew heaps are to leftist heaps as splay trees are to AVL trees
- Skew merge same as leftist merge, except we <u>always</u> swap left and right subheaps
- No need to maintain or test NPL of nodes
- Worst case is O(N)

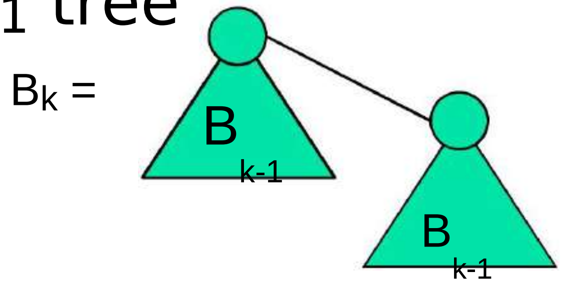
- Amortized cost of M operations is O(M log N)

Binomial Queues

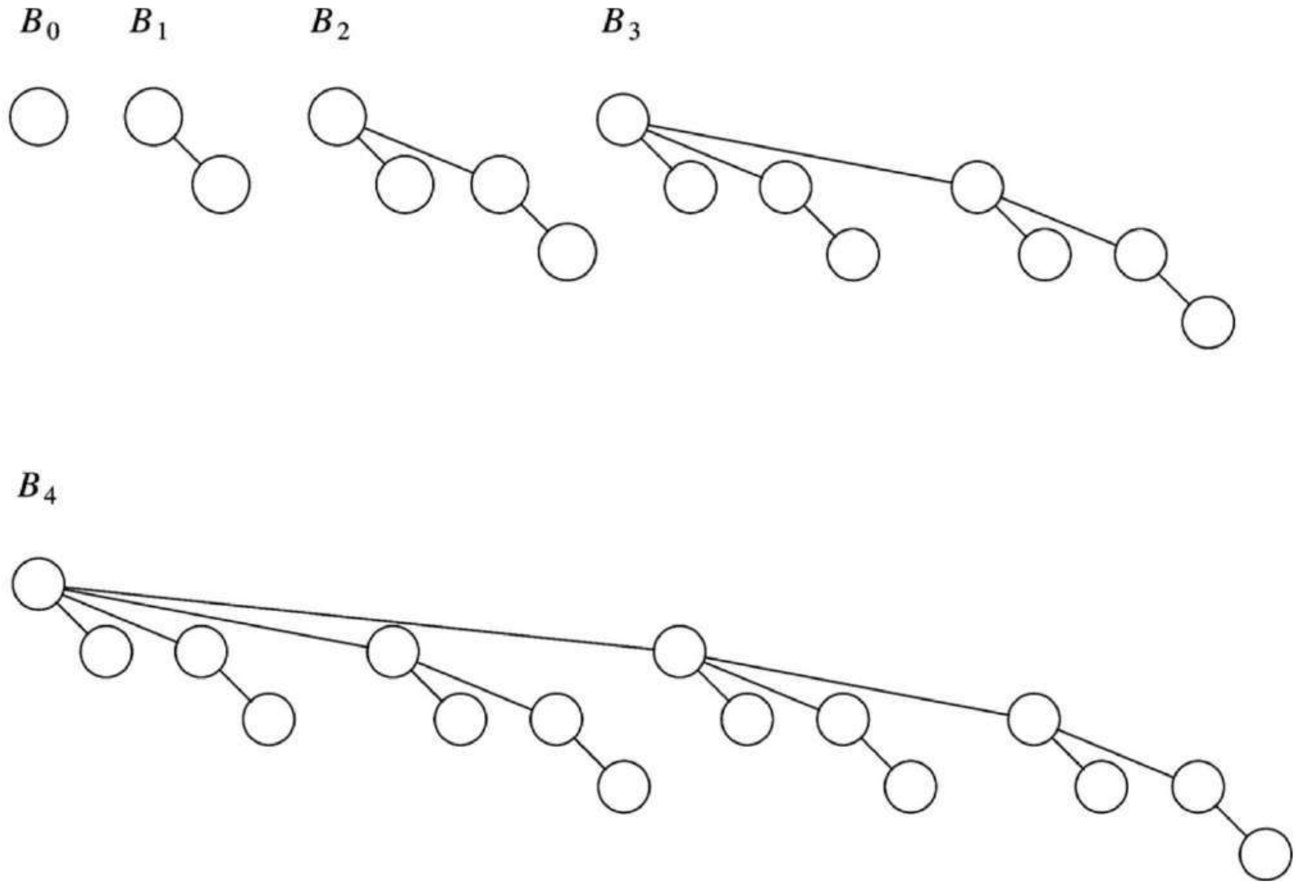
- Support all three operations in O(log N) worst-case time per operation
- Insertions take O(1) average-case time
- Key idea
  - Keep a collection of heap-ordered trees to postpone merging

Binomial Queues

- A binomial queue is a forest of binomial trees
  - Each in heap order
  - Each of a different height
- A binomial tree B_k of height k consists of two B_{k-1} binomial trees
  - The root of one B_{k-1} tree is the child of the root of the other B_{k-1} tree

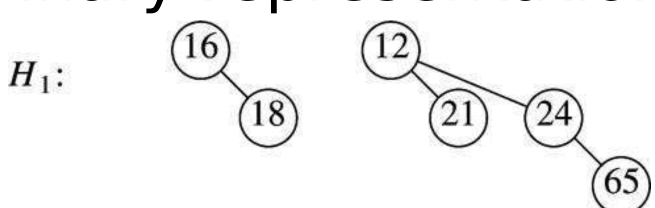


Binomial Trees



Binomial Trees

- Binomial trees of height k have exactly 2^k nodes
  - Number of nodes at depth d is $\binom{k}{d}$, the binomial coefficient
- A priority queue of any size can be represented by a binomial queue
  - Binary representation of B_k



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3

Binomial Queue Operations

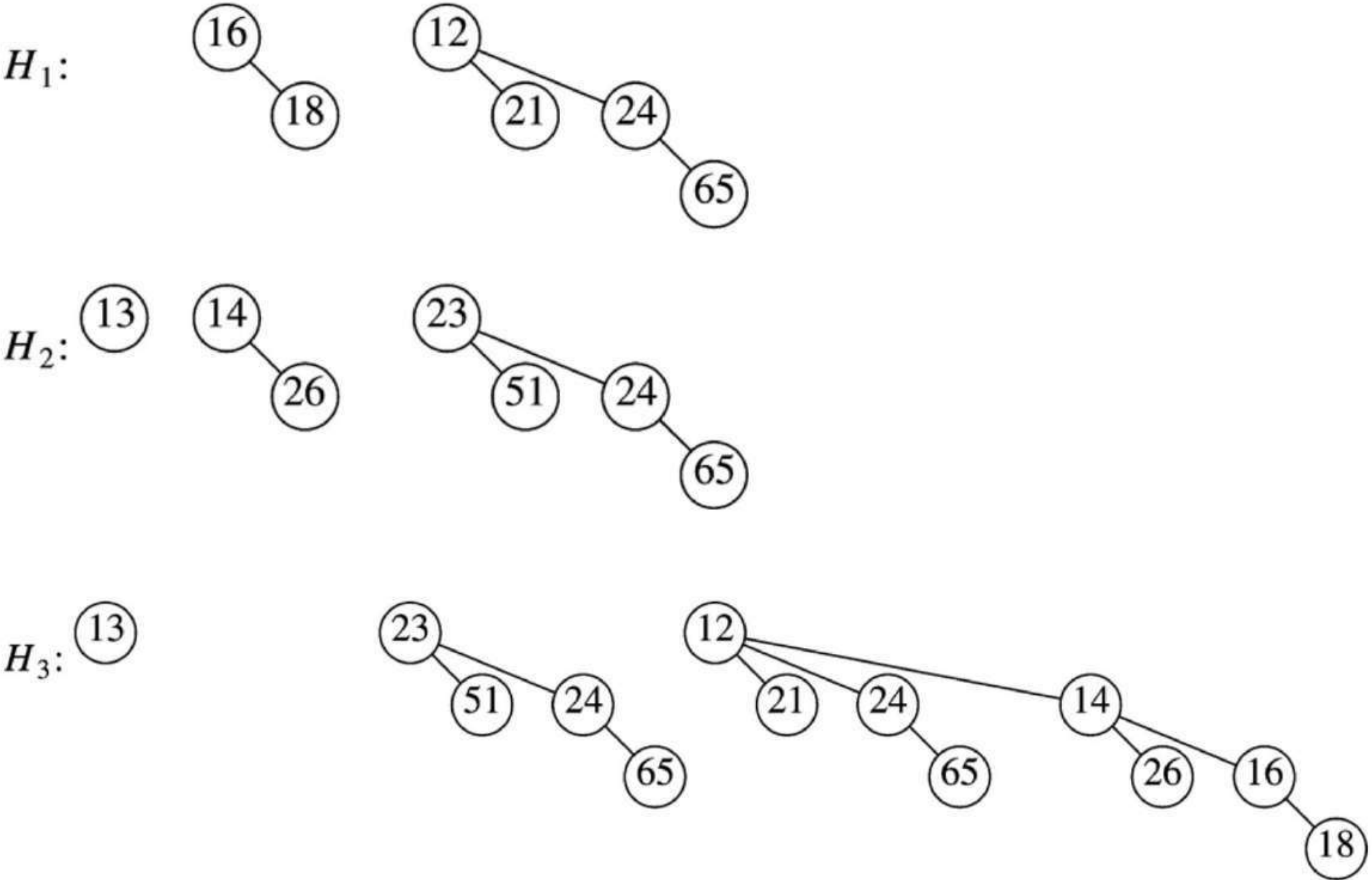
- Minimum element found by checking roots of all trees
  - At most ($\log_2 N$) of them, thus $O(\log N)$
  - Or, $O(1)$ by maintaining pointer to minimum element

4
4

Binomial Queue Operations

- Merge (H1,H2) → H3
  - Add trees of H1 and H2 into H3 in increasing order by depth
  - Traverse H3
    - If find two consecutive B_k trees, then create a B_{k+1} tree
    - If three consecutive B_k trees, then leave first, combine last two
    - Never more than three consecutive B_k trees
- Keep binomial trees ordered by height
- min(H3) = min(min(H1),min(H2))
- Running time O(log N)

Merge Example



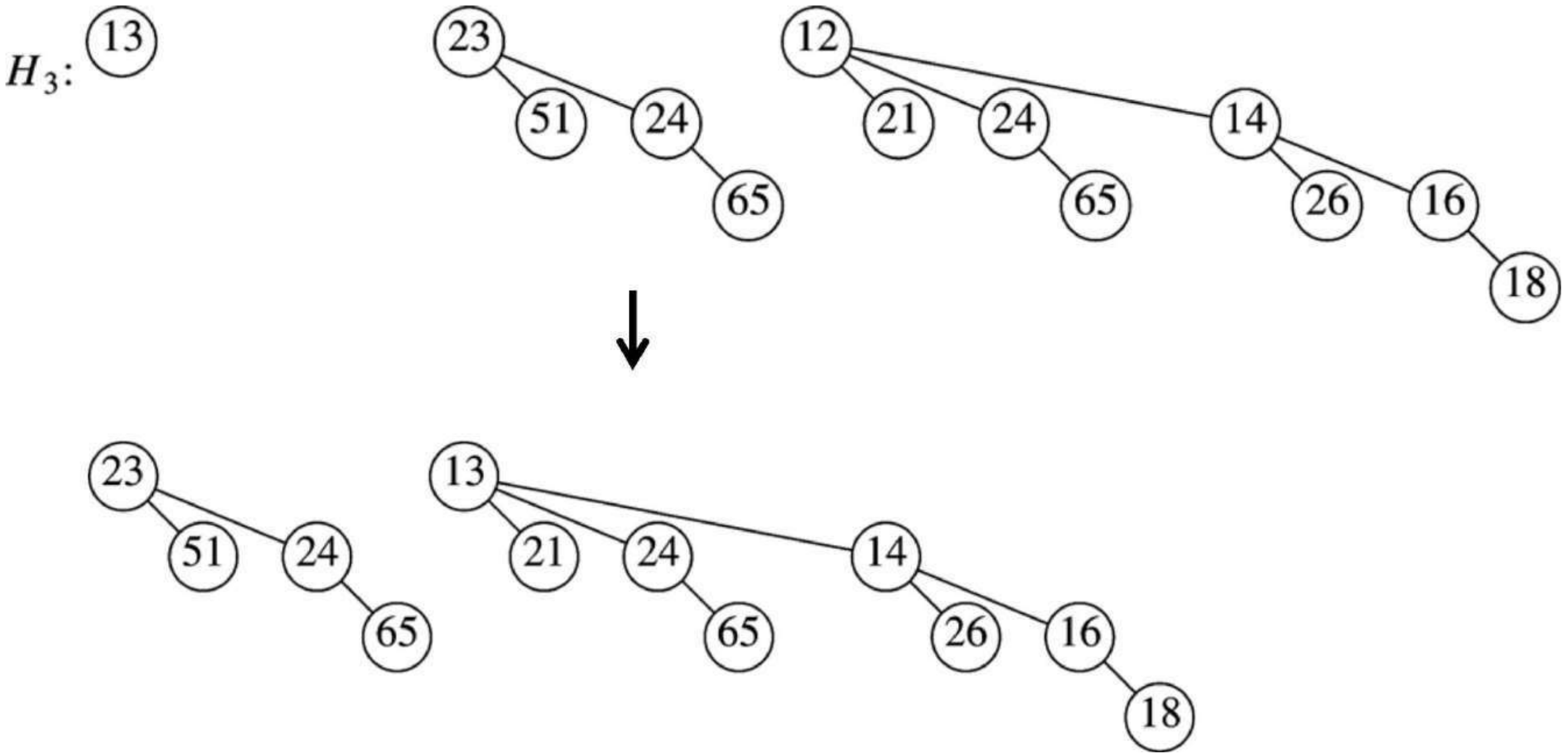
Binomial Queue Operations

- Insert (x, H1)
  - Create single-element queue H2
  - Merge (H1,H2)
- Running time proportional to minimum k such that B_k not in heap
- O(log N) worst case
- Probability B_k not present is 0.5
  - Thus, likely to find empty B_k after two tries on average
  - O(1) average case

Binomial Queue Operations

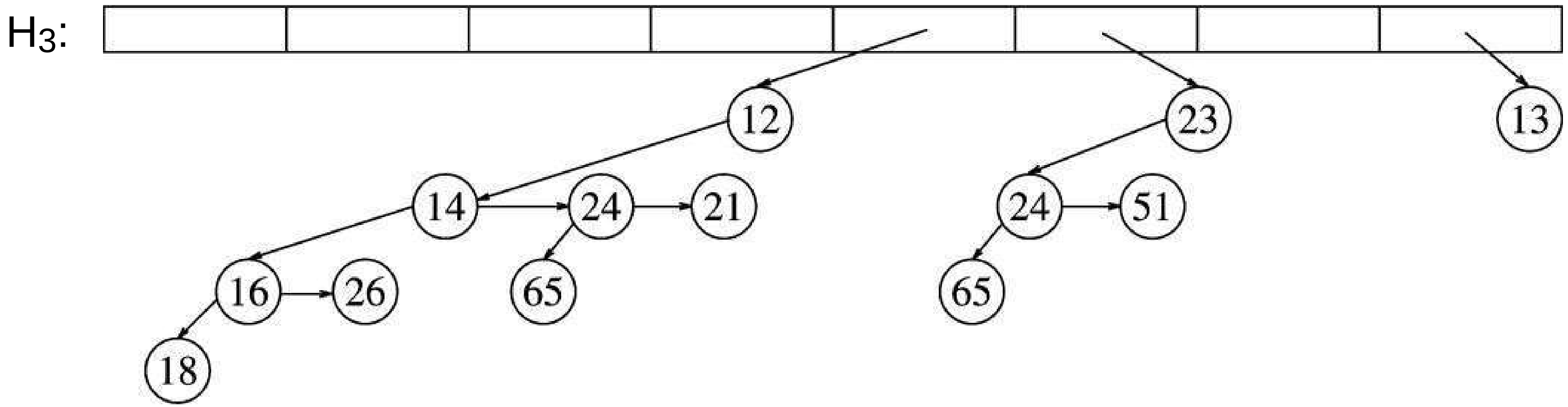
- deleteMin (H1)
  - Remove min(H1) tree from H1
  - Create heap H2 from the children of min(H)
  - Merge (H1,H2)
- Running time O(log N)

deleteMin Example



Binomial Queue Implementation

- Array of binomial trees
- Trees use first-child, right-sibling representation



```
template <typename Comparable>
class BinomialQueue
{
  public:
    BinomialQueue( );
    BinomialQueue( const Comparable & item );
    BinomialQueue( const BinomialQueue & rhs );
    ~BinomialQueue( );

    bool isEmpty( ) const;
    const Comparable & findMin( ) const;

    void insert( const Comparable & x );
    void deleteMin( );
    void deleteMin( Comparable & minItem );

    void makeEmpty( );
    void merge( BinomialQueue & rhs );

    const BinomialQueue & operator= ( const BinomialQueue & rhs );

```

```
  private:
    struct BinomialNode
    {
        Comparable    element;
        BinomialNode *leftChild;
        BinomialNode *nextSibling;

        BinomialNode( const Comparable & theElement,
                        BinomialNode *lt, BinomialNode *rt )
          : element( theElement ), leftChild( lt ), nextSibling( rt ) { }
    };

    enum { DEFAULT_TREES = 1 };

    int currentSize;                  // Number of items in priority queue
    vector<BinomialNode *> theTrees;  // An array of tree roots

    int findMinIndex( ) const;
    int capacity( ) const;
    BinomialNode * combineTrees( BinomialNode *t1, BinomialNode *t2 );
    void makeEmpty( BinomialNode * & t );
    BinomialNode * clone( BinomialNode *t ) const;
};
```

```
/**
 * Return the result of merging equal-sized t1 and t2.
 */
BinomialNode * combineTrees( BinomialNode *t1, BinomialNode *t2 )
{
    if( t2->element < t1->element )
        return combineTrees( t2, t1 );
    t2->nextSibling = t1->leftChild;
    t1->leftChild = t2;
    return t1;
}
```

```
/**
 * Merge rhs into the priority queue.
 * rhs becomes empty. rhs must be different from this.
 */
void merge( BinomialQueue & rhs )
{
    if( this == &rhs )    // Avoid aliasing problems
        return;

    currentSize += rhs.currentSize;

    if( currentSize > capacity( ) )
    {
        int oldNumTrees = theTrees.size( );
        int newNumTrees = max( theTrees.size( ), rhs.theTrees.size( ) ) + 1;
        theTrees.resize( newNumTrees );
        for( int i = oldNumTrees; i < newNumTrees; i++ )
            theTrees[ i ] = NULL;
    }

```

merge (cont.)

```
    BinomialNode *carry = NULL;
    for( int i = 0, j = 1; j <= currentSize; i++, j *= 2 )
    {
        BinomialNode *t1 = theTrees[ i ];
        BinomialNode *t2 = i < rhs.theTrees.size( ) ? rhs.theTrees[ i ]
                                                    : NULL;
        int whichCase = t1 == NULL ? 0 : 1;
        whichCase += t2 == NULL ? 0 : 2;
        whichCase += carry == NULL ? 0 : 4;

        switch( whichCase )
        {
          case 0: /* No trees */
          case 1: /* Only this */
            break;
          case 2: /* Only rhs */
            theTrees[ i ] = t2;
            rhs.theTrees[ i ] = NULL;
            break;
          case 4: /* Only carry */
            theTrees[ i ] = carry;
            carry = NULL;
            break;
```

```
          case 3: /* this and rhs */
            carry = combineTrees( t1, t2 );
            theTrees[ i ] = rhs.theTrees[ i ] = NULL;
            break;
          case 5: /* this and carry */
            carry = combineTrees( t1, carry );
            theTrees[ i ] = NULL;
            break;
          case 6: /* rhs and carry */
            carry = combineTrees( t2, carry );
            rhs.theTrees[ i ] = NULL;
            break;
          case 7: /* All three */
            theTrees[ i ] = carry;
            carry = combineTrees( t1, t2 );
            rhs.theTrees[ i ] = NULL;
            break;
        }
    }

    for( int k = 0; k < rhs.theTrees.size( ); k++ )
        rhs.theTrees[ k ] = NULL;
    rhs.currentSize = 0;
}
```

merge (cont.)

```
/**
 * Remove the minimum item and place it in minItem.
 * Throws UnderflowException if empty.
 */
void deleteMin( Comparable & minItem )
{
    if( isEmpty( ) )
        throw UnderflowException( );

    int minIndex = findMinIndex( );
    minItem = theTrees[ minIndex ]->element;

```

```
        BinomialNode *oldRoot = theTrees[ minIndex ];
        BinomialNode *deletedTree = oldRoot->leftChild;
        delete oldRoot;

        // Construct H''
        BinomialQueue deletedQueue;
        deletedQueue.theTrees.resize( minIndex + 1 );
        deletedQueue.currentSize = ( 1 << minIndex ) - 1;
        for( int j = minIndex - 1; j >= 0; j-- )
        {
            deletedQueue.theTrees[ j ] = deletedTree;
            deletedTree = deletedTree->nextSibling;
            deletedQueue.theTrees[ j ]->nextSibling = NULL;
        }

        // Construct H'
        theTrees[ minIndex ] = NULL;
        currentSize -= deletedQueue.currentSize + 1;

        merge( deletedQueue );
    }
```

deleteMin (cont.)

```
    /**
     * Find index of tree containing the smallest item in the priority queue.
     * The priority queue must not be empty.
     * Return the index of tree containing the smallest item.
     */
    int findMinIndex( ) const
    {
        int i;
        int minIndex;

        for( i = 0; theTrees[ i ] == NULL; i++ )
            ;

        for( minIndex = i; i < theTrees.size( ); i++ )
            if( theTrees[ i ] != NULL &&
                theTrees[ i ]->element < theTrees[ minIndex ]->element )
                minIndex = i;

        return minIndex;
    }
```

Priority Queues in STL

- Binary heap
- Maintains maximum element
- Methods
  - Push, top, pop, empty, clear

```
#include <iostream>
#include <queue>
using namespace std;
int main ()
{
  priority_queue<int> Q;
  for (int i=0; i<100; i++)
    Q.push(i);
  while (! Q.empty())
  {

    cout << Q.top() << endl;
    Q.pop();
  }
}
```

STL priority queue

```
#include <iostream>
#include <vector>
#include <queue>
#include <functional>
#include <string>
using namespace std;

// Empty the priority queue and print its contents.
template <typename PriorityQueue>
void dumpContents( const string & msg, PriorityQueue & pq )
{
    cout << msg << ":" << endl;
    while( !pq.empty( ) )
    {
        cout << pq.top( ) << endl;
        pq.pop( );
    }
}

// Do some inserts and removes (done in dumpContents).
int main( )
{
    priority_queue<int>                               maxPQ;
    priority_queue<int,vector<int>,greater<int> > minPQ;

    minPQ.push( 4 ); minPQ.push( 3 ); minPQ.push( 5 );
    maxPQ.push( 4 ); maxPQ.push( 3 ); maxPQ.push( 5 );

    dumpContents( "minPQ", minPQ );     // 3 4 5
    dumpContents( "maxPQ", maxPQ );     // 5 4 3

    return 0;
}
```

Summary

- Priority queues maintain the minimum or maximum element of a set
- Support O(log N) operations worst-case
  - insert, deleteMin, merge
- Support O(1) insertions average case

- Many applications in support of other algorithms