

## UNIT-IV

### Recurrence Relation

#### Generating Functions:

In mathematics, a **generating function** is a formal power series in one indeterminate, whose coefficients encode information about a sequence of numbers  $a_n$  that is indexed by the natural numbers. Generating functions were first introduced by Abraham de Moivre in 1730, in order to solve the general linear recurrence problem. One can generalize to formal power series in more than one indeterminate, to encode information about arrays of numbers indexed by several natural numbers.

Generating functions are not functions in the formal sense of a mapping from a domain to a codomain; the name is merely traditional, and they are sometimes more correctly called **generating series**.

#### Ordinary generating function

The *ordinary generating function* of a sequence  $a_n$  is

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n.$$

When the term *generating function* is used without qualification, it is usually taken to mean an ordinary generating function.

If  $a_n$  is the probability mass function of a discrete random variable, then its ordinary generating function is called a probability-generating function.

The ordinary generating function can be generalized to arrays with multiple indices. For example, the ordinary generating function of a two-dimensional array  $a_{m,n}$  (where  $n$  and  $m$  are natural numbers) is

$$G(a_{m,n}; x, y) = \sum_{m,n=0}^{\infty} a_{m,n} x^m y^n.$$

**Example:**

$$G(n^2; x) = \sum_{n=0}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}$$

### Exponential generating function

The *exponential generating function* of a sequence  $a_n$  is

$$\text{EG}(a_n; x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

**Example:**

$$\text{EG}(n^2; x) = \sum_{n=0}^{\infty} \frac{n^2 x^n}{n!} = x(x+1)e^x$$

### Function of Sequences:

Generating functions giving the first few powers of the nonnegative integers are given in the following table.

| $n^p$ | $f(x)$                              | series                            |
|-------|-------------------------------------|-----------------------------------|
| 1     | $\frac{x}{1-x}$                     | $x + x^2 + x^3 + \dots$           |
| $n$   | $\frac{x}{(1-x)^2}$                 | $x + 2x^2 + 3x^3 + 4x^4 + \dots$  |
| $n^2$ | $\frac{x(x+1)}{(1-x)^3}$            | $x + 4x^2 + 9x^3 + 16x^4 + \dots$ |
| $n^3$ | $\frac{x(x^2+4x+1)}{(1-x)^4}$       | $x + 8x^2 + 27x^3 + \dots$        |
| $n^4$ | $\frac{x(x+1)(x^2+10x+1)}{(1-x)^5}$ | $x + 16x^2 + 81x^3 + \dots$       |

There are many beautiful generating functions for special functions in number theory. A few particularly nice examples are

$$f(x) = \frac{1}{(x)_{\infty}} \quad (2)$$

$$= \sum_{n=0}^{\infty} P(n) x^n \quad (3)$$

$$= 1 + x + 2x^2 + 3x^3 + \dots \quad (4)$$

for the partition function  $P$ , where  $(q)_\infty$  is a  $q$ -Pochhammer symbol, and

$$f(x) = \frac{x}{1-x-x^2} \quad (5)$$

$$= \sum_{n=0}^{\infty} F_n x^n \quad (6)$$

$$= x + x^2 + 2x^3 + 3x^4 + \dots \quad (7)$$

for the Fibonacci numbers  $F_n$ .

Generating functions are very useful in combinatorial enumeration problems. For example, the subset sum problem, which asks the number of ways to select  $m$  out of  $M$  given integers such that their sum equals  $s$ , can be solved using generating functions.

## Calculating Coefficient of generating function:

By using the following polynomial expansions, we can calculate the coefficient of a generating function.

### Polynomial Expansions:

$$1) \frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \dots + x^n$$

$$2) \frac{1}{1-x} = 1 + x + x^2 + \dots$$

$$3) (1+x)^n = 1 + C(n,1)x + C(n,2)x^2 + \dots + C(n,r)x^r + \dots + C(n,n)x^n$$

$$4) (1+x^m)^n = 1 + C(n,1)x^m + C(n,2)x^{2m} + \dots + C(n,k)x^{km} + \dots + C(n,n)x^{nm}$$

$$5) \frac{1}{(1-x)^n} = 1 + C(n-1,1)x + C(n,2)x^2 + \dots + C(n-1,r)x^r + \dots$$

$$6) \text{ If } h(x) = f(x)g(x), \text{ where } f(x) = a_0 + a_1x + a_2x^2 + \dots \text{ and } g(x) = b_0 + b_1x + b_2x^2 + \dots, \text{ then}$$

$$h(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots + (a_0b_r + a_1b_{r-1} + \dots + a_{r-1}b_1 + a_rb_0)x^r + \dots$$

## Example

Find the coefficient of  $x^{16}$  in  $(x^2 + x^3 + x^4 + \dots)^5$

$x^{16}$  in  $x^{10}(1-x)^{-5}$  [i.e., the  $x^6$  term in  $(1-x)^{-5}$  is

*multiplied by  $x^{10}$  to become the  $x^{16}$  term in  $x^{10}(1-x)^{-5}$ ]*

To simplify the expression, we extract  $x^2$  from each polynomial factor and then apply identity (2).

$$\begin{aligned}(x^2 + x^3 + x^4 + \dots)^5 &= [x^2(1 + x + x^2 + \dots)]^5 \\ &= x^{10}(1 + x + x^2 + \dots)^5 \\ &= x^{10} \frac{1}{(1-x)^5}\end{aligned}$$

Thus the coefficient of  $x^{16}$  in  $(x^2 + x^3 + x^4 + \dots)^5$  is the coefficient of  $x^{16}$  in  $x^{10}(1-x)^{-5}$  [i.e., the  $x^6$  term in  $(1-x)^{-5}$  is multiplied by to become the  $x^{16}$  term in  $x^{10}(1-x)^{-5}$ ]

$$\frac{1}{(1-x)^n} = 1 + C(1+n-1, 1)x + C(2+n-1, 2)x^2 + \dots + C(r+n-1, r)x^r + \dots$$

From expansion (5) we see that the coefficient of  $x^6$  in  $(1-x)^{-5}$  is  $C(6+5-1, 6)$

More generally, the coefficient of  $x^r$  in

$x^r$  in  $x^{10}(1-x)^{-5}$  equals the coefficient of  $x^{r-10}$  in  $(1-x)^{-5}$ , namely,  $C((r-10)+5-1, (r-10))$ .

## Recurrence relations:

**Introduction :** A recurrence relation is a formula that relates for any integer  $n \geq 1$ , the  $n$ -th term of a sequence  $A = \{a_r\}_{r=0}$  to one or more of the terms  $a_0, a_1, \dots, a_{n-1}$ . Example. If  $S_n$  denotes the sum of the first  $n$  positive integers, then

10.  $S_n = n + S_{n-1}$ . Similarly if  $d$  is a real number, then the  $n$ th term of an arithmetic progression with common difference  $d$  satisfies the relation

11.  $a_n = a_{n-1} + d$ . Likewise if  $p_n$  denotes the  $n$ th term of a geometric progression with common ratio  $r$ , then

■  $p_n = r p_{n-1}$ . We list other examples

$$\text{as: } a_n - 3a_{n-1} + 2a_{n-2} = 0.$$

$$a_n - 3a_{n-1} + 2a_{n-2} = n^2 + 1.$$

$$a_n - (n-1)a_{n-1} - (n-1)a_{n-2} = 0.$$

$$a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 5n.$$

$$a_n - 3(a_{n-1})^2 + 2a_{n-2} = n.$$

$$a_n = a_0 a_{n-1} + a_1 a_{n-2} + \dots + a_{n-1} a_0.$$

$$a_{2n} + (a_{n-1})^2 = -1.$$

**Definition.** Suppose  $n$  and  $k$  are nonnegative integers. A recurrence relation of the form  $c_0(n)a_n + c_1(n)a_{n-1} + \dots + c_k(n)a_{n-k} = f(n)$  for  $n \geq k$ , where  $c_0(n), c_1(n), \dots, c_k(n)$ , and  $f(n)$  are functions of  $n$  is said to be a **linear recurrence relation**. If  $c_0(n)$  and  $c_k(n)$  are not identically zero, then it is said to be a **linear recurrence relation degree  $k$** . If  $c_0(n), c_1(n), \dots, c_k(n)$  are constants, then the recurrence relation is known as a **linear relation with constant coefficients**. If  $f(n)$  is identically zero, then the recurrence relation is said to be **homogeneous**; otherwise, it is **inhomogeneous**.

Thus, all the examples above are linear recurrence relations except (8), (9), and (10); the relation (8), for instance, is not linear because of the squared term.

The relations in (3), (4), (5), and (7) are linear with constant coefficients.

Relations (1), (2), and (3) have degree 1; (4), (5), and (6) have degree 2; (7) has degree 3. Relations (3), (4), and (6) are homogeneous.

There are no general techniques that will enable one to solve all recurrence relations. There are, nevertheless, techniques that will enable us to solve linear recurrence relations with constant coefficients.

## SOLVING RECURRENCE RELATIONS BY SUBSTITUTION AND GENERATING FUNCTIONS

We shall consider four methods of solving recurrence relations in this and the next two sections:

5. Substitution (also called iteration),
6. Generating functions,
7. Characteristics roots, and
8. Undetermined coefficients.

In the substitution method the recurrence relation for  $a_n$  is used repeatedly to solve for a general expression for  $a_n$  in terms of  $n$ . We desire that this expression involve no other terms of the sequence except those given by boundary conditions.

The mechanics of this method are best described in terms of examples. We used this method in Example 5.3.4. Let us also illustrate the method in the following examples.

**Example**

Solve the recurrence relation  $a_n = a_{n-1} + f(n)$  for  $n \geq 1$  by substitution

$$a_1 = a_0 + f(1)$$

$$a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$$

$$a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3)$$

$$\vdots$$

$$a_n = a_0 + f(1) + f(2) + \dots + f(n)$$

$$= a_0 + \sum_{k=1}^n f(k)$$

Thus,  $a_n$  is just the sum of the  $f(k)$ 's plus  $a_0$ .

More generally, if  $c$  is a constant then we can solve  $a_n = c a_{n-1} + f(n)$  for  $n \geq 1$  in the same way:

$$a_1 = c a_0 + f(1)$$

$$a_2 = c a_1 + f(2) = c(c a_0 + f(1)) + f(2) = c^2 a_0 + c f(1) + f(2)$$

$$a_3 = c a_2 + f(3) = c(c^2 a_0 + c f(1) + f(2)) + f(3) = c^3 a_0 + c^2 f(1) + c f(2) + f(3)$$

$$\vdots$$

$$a_n = c a_{n-1} + f(n) = c(c^{n-1} a_0 + c^{n-2} f(1) + \dots + c^{n-2} f(n-1)) + f(n) = c^n a_0 + c^{n-1} f(1) + c^{n-2} f(2) + \dots + c f(n-1) + f(n)$$

Or

$$a_n = c^n a_0 + \sum_{k=1}^n c^{n-k} f(k)$$

**Solution of Linear Inhomogeneous Recurrence Relations:**

The equation  $a_n + c_1 a_{n-1} + c_2 a_{n-2} = f(n)$ , where  $c_1$  and  $c_2$  are constant, and  $f(n)$  is not identically 0, is called a second-order linear **inhomogeneous** recurrence relation (or difference equation) with constant coefficients. The homogeneous case, which we've looked at already, occurs when

$f(n) \equiv 0$ . The inhomogeneous case occurs more frequently. The homogeneous case is so important largely because it gives us the key to solving the inhomogeneous equation. If you've studied linear differential equations with constant coefficients, you'll see the parallel. We will call the

difference obtained by setting the right-hand side equal to 0, the “associated homogeneous equation.” We know how to solve this. Say that  $V$  is a solution. Now suppose that  $(n)$  is any particular solution of the inhomogeneous equation. (That is, it solves the equation, but does not necessarily match the initial data.) Then  $U = V + (n)$  is a solution to the inhomogeneous equation, which you can see simply by substituting  $U$  into the equation. On the other hand, every solution  $U$  of the inhomogeneous equation is of the form  $U = V + (n)$  where  $V$  is a solution of the homogeneous equation, and  $g(n)$  is a particular solution of the inhomogeneous equation. The proof of this is straightforward. If we have two solutions to the inhomogeneous equation, say  $U_1$  and  $U_2$ , then their difference  $U_1 - U_2 = V$  is a solution to the homogeneous equation, which you can check by substitution. But then  $U_1 = V + U_2$ , and we can set  $U_2 = (n)$ , since by assumption,  $U_2$  is a particular solution. This leads to the following theorem: **the general solution to the inhomogeneous equation is the general solution to the associated homogeneous equation, plus any particular solution to the inhomogeneous equation.** This gives the following procedure for solving the inhomogeneous equation:

4. Solve the associated homogeneous equation by the method we’ve learned. This will involve variable (or undetermined) coefficients.
5. Guess a particular solution to the inhomogeneous equation. It is because of the guess that I’ve called this a procedure, not an algorithm. For simple right-hand sides, we can say how to compute a particular solution, and in these cases, the procedure merits the name “algorithm.”
6. The general solution to the inhomogeneous equation is the sum of the answers from the two steps above.
7. Use the initial data to solve for the undetermined coefficients from step 1.

To solve the equation  $an - 6an-1 + 8an-2 = 3$ . Let’s suppose that we are also given the initial data  $a_0 = 3$ ,  $a_1 = 3$ . The associated homogeneous equation is  $an - 6an-1 + 8an-2 = 0$ , so the characteristic equation is  $r^2 - 6r + 8 = 0$ , which has roots  $r_1 = 2$  and  $r_2 = 4$ . Thus, the general solution to the associated homogeneous equation is  $c_1 2^n + c_2 4^n$ . When the right-hand side is a polynomial, as in this case, there will always be a particular solution that is a polynomial.

Usually, a polynomial of the same degree will work, so we’ll guess in this case that there is a constant  $C$  that solves the homogeneous equation. If that is so, then  $an = an-1 = an-2 = C$ , and substituting into the equation gives  $C - 6C + 8C = 3$ , and we find that  $C = 1$ . Now, the general solution to the inhomogeneous equations is  $c_1 2^n + c_2 4^n + 1$ . Reassuringly, this is the answer given in the back of the book. Our initial data lead to the equations  $c_1 + c_2 + 1 = 3$  and  $2c_1 + 4c_2 + 1 = 3$ , whose solution is  $c_1 = 3$ ,  $c_2 = -1$ . Finally, the solution to the inhomogeneous equation, with the initial condition given, is  $an = 3 \cdot 2^n - 4^n + 1$ . Sometimes, a polynomial of the same degree as the right-hand side doesn’t work. This happens when the characteristic equation has 1 as a root. If our equation had been  $an - 6an-1 + 5an-2 = 3$ , when we guessed that the particular solution was a constant  $C$ , we’d have arrived at the equation  $C - 6C + 5C = 3$ , or  $0 = 3$ . The way to deal with this is to increase the degree of the polynomial. Instead of assuming that the solution is constant, we’ll assume that it’s linear. In fact, we’ll guess that it is of the form